

# G<sub>2</sub>-MANIFOLDS WITH PARALLEL CHARACTERISTIC TORSION

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**ABSTRACT.** We classify 7-dimensional cocalibrated G<sub>2</sub>-manifolds with parallel characteristic torsion and non-abelian holonomy. All these spaces admit a metric connection  $\nabla^c$  with totally skew-symmetric torsion and a spinor field  $\Psi_1$  solving the equations in the common sector of type II superstring theory. There exist G<sub>2</sub>-structures with parallel characteristic torsion that are not naturally reductive.

## 1. Metric connections with parallel torsion

Consider a Riemannian manifold and denote by  $\nabla^g$  its Levi-Civita connection. Any 3-form  $T$  defines via the formula

$$\nabla_X Y := \nabla_X^g Y + \frac{1}{2} T(X, Y, *)$$

a metric connection  $\nabla$  with totally skew-symmetric torsion  $T$ . We are interested in the case that the torsion form is parallel,  $\nabla T = 0$ . Then  $T$  is coclosed,  $\delta(T) = 0$ , and the differential  $dT$  depends only on the algebraic type of  $T$  (see [13]),

$$dT = \sum_{i=1}^n (e_i \lrcorner T) \wedge (e_i \lrcorner T) .$$

The curvature tensor  $R^\nabla : \Lambda^2 \rightarrow \Lambda^2$  of the connection  $\nabla$  is symmetric,  $R^\nabla(X, Y, U, V) = R^\nabla(U, V, X, Y)$ , see [13]. Moreover, if there exists a  $\nabla$ -parallel spinor field  $\Psi$ , we can compute the Ricci tensor directly using only the torsion form,

$$2 \operatorname{Ric}^\nabla(X) \cdot \Psi = (X \lrcorner dT) \cdot \Psi .$$

In particular  $\operatorname{Ric}^\nabla$  is parallel and divergence free,  $\nabla \operatorname{Ric}^\nabla = 0$ ,  $\operatorname{div}(\operatorname{Ric}^\nabla) = 0$  (see [3]). In [2] we introduced a second order differential operator  $\Omega$  acting on spinors. It is a generalization of the Casimir operator and its kernel contains all  $\nabla$ -parallel spinors. The formula simplifies for parallel torsion, yielding

$$\Omega = \Delta_T + \frac{1}{16}(2 \operatorname{Scal}^g + \|T\|^2) - \frac{1}{4} T^2 .$$

Consequently, any  $\nabla$ -parallel spinor  $\Psi$  satisfies

$$T^2 \cdot \Psi = \frac{1}{4}(2 \operatorname{Scal}^g + \|T\|^2) \cdot \Psi .$$

In case we have more than one  $\nabla$ -parallel spinor, the latter equation is an algebraic restriction for the 3-form  $T$ . Indeed, the endomorphism  $T^2$  acts as a scalar on the space of all  $\nabla$ -parallel spinors.

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Almost Hermitian manifolds with parallel characteristic torsion have been studied in [4] and [17]. In this paper we consider 7-dimensional Riemannian manifolds equipped with a  $G_2$ -structure  $\varphi$ . It is called *cocalibrated* if the 3-form  $\varphi$  satisfies the differential equation

$$d * \varphi = 0.$$

This ensures the existence of a characteristic connection  $\nabla^c$  with totally skew-symmetric torsion preserving the  $G_2$ -structure (see [13])

$$T^c = \frac{1}{6} (d\varphi, * \varphi) \cdot \varphi - * d\varphi.$$

In general, the torsion form  $T^c$  is not  $\nabla^c$ -parallel. If it is, there exists a spinor field  $\Psi_1$  satisfying the equations

$$\nabla^c \Psi_1 = 0, \quad T^c \cdot \Psi_1 = a \cdot \Psi_1, \quad \delta(T^c) = 0, \quad \nabla^c \text{Ric}^{\nabla^c} = 0,$$

where the factor  $a$  depends on the algebraic type of the torsion form. Consequently,  $G_2$ -manifolds with a parallel characteristic torsion are solutions of the equations for the common sector of type II superstring theory, see [3]. The holonomy algebra  $\mathfrak{hol}(\nabla^c) \subset \mathfrak{g}_2$  preserves a 3-form  $T^c$ . The aim of the present paper is the construction and classification of cocalibrated  $G_2$ -manifolds with parallel characteristic torsion and non-abelian holonomy algebra. There are eight non-abelian subalgebras of  $\mathfrak{g}_2$ , see [10]. For any of these algebras, we describe the set of admissible torsion forms  $T^c$ . We discuss the geometry of the space  $M^7$  in dependence of the type of its holonomy algebra as well as the  $G_2$ -orbit of the torsion form  $T^c \in \Lambda_1^3 \oplus \Lambda_{27}^3$ . In particular, two subalgebras of  $\mathfrak{g}_2$  are realized as holonomy algebras of a unique cocalibrated  $G_2$ -manifold with parallel characteristic torsion, see Theorem 8.1 and Theorem 9.1. All other non-abelian subalgebras occur as holonomy algebras for whole families of cocalibrated  $G_2$ -manifolds.

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## 2. The exceptional Lie algebra $\mathfrak{g}_2$ and its subalgebras

The group  $G_2$  is a compact, simple and simply-connected 14-dimensional Lie group. It consists of all elements in the group  $SO(7)$  preserving the 3-form in seven dimensions

$$\varphi = e_{127} + e_{135} - e_{146} - e_{236} - e_{245} + e_{347} + e_{567}.$$

The group  $G_2$  lifts into the spin group  $\text{Spin}(7)$  and, consequently, it acts on spinors. Let  $e_1, \dots, e_7$  be the standard orthonormal basis of Euclidian space  $\mathbb{R}^7$  and denote by  $\text{Cliff}(\mathbb{R}^7)$  the real Clifford algebra. We will use the following real representation of this algebra on the space of real spinors  $\Delta_7 := \mathbb{R}^8$ :

$$\begin{aligned} e_1 &= E_{18} + E_{27} - E_{36} - E_{45}, & e_2 &= -E_{17} + E_{28} + E_{35} - E_{46}, \\ e_3 &= -E_{16} + E_{25} - E_{38} + E_{47}, & e_4 &= -E_{15} - E_{26} - E_{37} - E_{48}, \\ e_5 &= -E_{13} - E_{24} + E_{57} + E_{68}, & e_6 &= E_{14} - E_{23} - E_{58} + E_{67}, \\ e_7 &= E_{12} - E_{34} - E_{56} + E_{78} \end{aligned}$$

where  $E_{ij}$  denotes the standard basis of the Lie algebra  $\mathfrak{so}(8)$ . We fix an orthonormal basis  $\psi_1 := (1, 0, \dots, 0), \dots, \psi_8 := (0, 0, \dots, 1)$  of spinors. The group  $G_2$  coincides with the subgroup of  $\text{Spin}(7)$  preserving the spinor  $\psi_1$ . Its Lie algebra  $\mathfrak{g}_2$  is the subalgebra

of  $\mathfrak{spin}(7)$  containing all forms  $\omega = \sum \omega_{ij} e_{ij}$  such that the Clifford product  $\omega \cdot \psi_1 = 0$  vanishes. This condition holds if and only if

$$\begin{aligned} \omega_{12} + \omega_{34} + \omega_{56} &= 0, & -\omega_{13} + \omega_{24} - \omega_{67} &= 0, & -\omega_{14} - \omega_{23} - \omega_{57} &= 0, \\ -\omega_{16} - \omega_{25} + \omega_{37} &= 0, & \omega_{15} - \omega_{26} - \omega_{47} &= 0, & \omega_{17} + \omega_{36} + \omega_{45} &= 0, \\ \omega_{27} + \omega_{35} - \omega_{46} &= 0. \end{aligned}$$

These equations define the Lie algebra  $\mathfrak{g}_2$  inside  $\mathfrak{spin}(7)$ . The representations of  $\mathfrak{g}_2$  on  $\mathbb{R}^7$  and on  $\text{Lin}(\psi_2, \psi_3, \dots, \psi_8)$  are equivalent.

Dynkin's classical result that  $\mathfrak{g}_2$  has exactly three maximal subalgebras of dimensions 8, 6 and 3 respectively implies that there are (up to conjugation) eight different non-abelian subalgebras of  $\mathfrak{g}_2$  (see [10]). The  $G_2$ -representation  $\Lambda^3(\mathbb{R}^7)$  splits into a 1-dimensional trivial representation  $\Lambda_1^3$  generated by the form  $\varphi$ , a 7-dimensional representation  $\Lambda_7^3$  containing all inner products  $e_i \lrcorner * \varphi$ , and the 27-dimensional complement  $\Lambda_{27}^3$ . Given a Lie subalgebra  $\mathfrak{g} \subset \mathfrak{g}_2$  let  $(\Delta_7)_{\mathfrak{g}}$  and  $(\Lambda_{27}^3)_{\mathfrak{g}}$  be the space of all  $\mathfrak{g}$ -invariant spinors and the space of all  $\mathfrak{g}$ -invariant 3-forms in  $\Lambda_{27}^3$ , respectively. The space of torsion forms of cocalibrated  $G_2$ -manifolds with parallel torsion is the following set,

$$\text{Tor}_{\mathfrak{g}} := \{T \in \Lambda_1^3 \oplus (\Lambda_{27}^3)_{\mathfrak{g}} : T^2 \text{ acts as a scalar on } (\Delta_7)_{\mathfrak{g}}\}.$$

Since the  $G_2$ -orbit of the characteristic torsion is a geometric invariant, two torsion forms  $T_1$  and  $T_2$  define equivalent geometric structures if they are equivalent as 3-forms under the action of the normalizer  $\mathcal{N}(G)$  of the Lie group  $G \subset G_2$ . Consequently, the relevant set is  $\text{Tor}_{\mathfrak{g}}/\mathcal{N}(G)$ . We computed these spaces for any of the eight non-abelian subalgebras of  $\mathfrak{g}_2$ . Here we formulate only the result of these computations.

### The subalgebra $\mathfrak{su}(3) \subset \mathfrak{g}_2$ .

The Lie algebra preserving two spinors  $\psi_1$  and  $\psi_2$  is isomorphic to  $\mathfrak{su}(3)$ . It is the unique subalgebra of  $\mathfrak{g}_2$  of dimension eight. The representation on Euclidian space splits into  $\mathbb{R}^7 = \mathbb{C}^3 \oplus \mathbb{R}^1$ .  $(\Lambda_{27}^3)_{\mathfrak{su}(3)}$  is one-dimensional and generated by

$$T = 4e_{127} - 3e_{135} + 3e_{146} + 3e_{236} + 3e_{245} + 4e_{347} + 4e_{567}.$$

The set  $\text{Tor}_{\mathfrak{su}(3)}$  is the union of two lines,

$$\text{Tor}_{\mathfrak{su}(3)} = \{a \cdot (e_{127} + e_{347} + e_{567})\} \cup \{b \cdot (-e_{135} + e_{146} + e_{236} + e_{245})\}.$$

Remark that the intersections  $\text{Tor}_{\mathfrak{su}(3)} \cap \Lambda_1^3 = 0$  and  $\text{Tor}_{\mathfrak{su}(3)} \cap \Lambda_{27}^3 = 0$  are trivial.

### The subalgebra $\mathfrak{u}(2) \subset \mathfrak{su}(3) \subset \mathfrak{g}_2$ .

This Lie subalgebra is generated by

$$P_1 := e_{13} + e_{24}, \quad P_2 := e_{14} - e_{23}, \quad P_3 := e_{12} - e_{34}, \quad Q_3 := e_{12} + e_{34} - 2e_{56}.$$

The subalgebra  $\mathfrak{u}(2) \subset \mathfrak{g}_2$  preserves two spinors  $\psi_1, \psi_2$  and it acts on Euclidian space as  $\mathbb{R}^7 = \mathbb{C}^2 \oplus \mathbb{C} \oplus \mathbb{R}^1$ . The space  $(\Lambda_{27}^3)_{\mathfrak{u}(2)}$  is two-dimensional and parameterized by

$$T_{a,b} := (a + b)(-e_{135} + e_{146} + e_{236} + e_{245}) + 2a(e_{347} + e_{127}) + 4be_{567}.$$

$\text{Tor}_{\mathfrak{u}(2)}$  is the union of two planes in  $\Lambda_1^3 \oplus (\Lambda_{27}^3)_{\mathfrak{u}(2)}$ . The first plane  $\Pi_1$  is parameterized by

$$T_{a,b} + c \cdot \varphi, \quad \text{where } a + b = c.$$

The second plane  $\Pi_2$  is the family

$$T_{a,b} + c \cdot \varphi, \quad \text{where} \quad 4(a+b) = -3c.$$

Here the intersection  $\Pi_1 \cap \Pi_2$  is a line in  $(\Lambda_{27}^3)_{\mathfrak{u}(2)}$ .

**The subalgebra  $\mathfrak{su}(2) \subset \mathfrak{su}(3) \subset \mathfrak{g}_2$ .**

This Lie subalgebra is generated by  $P_1, P_2, P_3$  and stabilizes four spinors  $\psi_1, \psi_2, \psi_3, \psi_4$ . Under  $\mathfrak{su}(2)$ , we have the decomposition  $\mathbb{R}^7 = \mathbb{C}^2 \oplus \mathbb{R}^1 \oplus \mathbb{R}^1 \oplus \mathbb{R}^1$ . It turns out that  $(\Lambda_{27}^3)_{\mathfrak{su}(2)}$  is a 6-dimensional space. Moreover, the normalizer  $\mathcal{N}(\text{SU}(2))$  is a three-dimensional subgroup. It acts on the spinor  $\psi_2, \psi_3, \psi_4$  in a non-trivial way. Consequently, the geometrically relevant set  $\text{Tor}_{\mathfrak{su}(2)}/\mathcal{N}(\text{SU}(2))$  consists of 3-forms admitting the four spinors  $\psi_1, \psi_2, \psi_3, \psi_4$  as eigenspinors. It is the union of eight lines in  $\Lambda_1^3 \oplus (\Lambda_{27}^3)_{\mathfrak{su}(2)}$  generated by the following forms:

$$\begin{aligned} T &= e_{127} - e_{135} + e_{146} + e_{236} + e_{245} + e_{347} - 2e_{567}, \\ T &= e_{127} + e_{135} - e_{146} - e_{236} - e_{245} + e_{347} - 2e_{567}, \\ T &= e_{127} + e_{135} + e_{146} + e_{236} - e_{245} + e_{347} + 2e_{567}, \\ T &= e_{127} - e_{135} - e_{146} - e_{236} + e_{245} + e_{347} + 2e_{567}, \\ T &= e_{135} - e_{245}, \quad T = e_{146} + e_{236}, \quad T = e_{127} + e_{347}, \quad T = e_{567}. \end{aligned}$$

The intersections of  $\text{Tor}_{\mathfrak{su}(2)} \cap \Lambda_1^3 = 0$  and  $\text{Tor}_{\mathfrak{su}(2)} \cap \Lambda_{27}^3 = 0$  are trivial.

**The subalgebra  $\mathfrak{su}_c(2) \subset \mathfrak{g}_2$ .**

The centralizer of the subalgebra  $\mathfrak{su}(2) \subset \mathfrak{g}_2$  is a subalgebra of  $\mathfrak{g}_2$  which is isomorphic, but not conjugated to  $\mathfrak{su}(2)$ . We denote this algebra by  $\mathfrak{su}_c(2)$ . It is generated by

$$Q_1 := -e_{14} - e_{23} + 2e_{57}, \quad Q_2 := -e_{13} + e_{24} + 2e_{67}, \quad Q_3 := e_{12} + e_{34} - 2e_{56}.$$

The subalgebra  $\mathfrak{su}_c(2) \subset \mathfrak{g}_2$  preserves only one spinor  $\psi_1$  and Euclidian space splits under its action into  $\mathbb{R}^7 = \mathbb{C}^2 \oplus \mathbb{R}^3$ . The space  $(\Lambda_{27}^3)_{\mathfrak{su}_c(2)}$  is one-dimensional and generated by the 3-form

$$T = \varphi - 7e_{567}.$$

The set  $\text{Tor}_{\mathfrak{su}_c(2)}$  coincides with  $\Lambda_1^3 \oplus (\Lambda_{27}^3)_{\mathfrak{su}_c(2)}$  and is generated by  $\varphi$  and  $e_{567}$ .

**The subalgebra  $\mathbb{R}^1 \oplus \mathfrak{su}_c(2) \subset \mathfrak{g}_2$ .**

This subalgebra is generated by  $P_1, Q_1, Q_2, Q_3$ . The subalgebra  $\mathbb{R}^1 \oplus \mathfrak{su}_c(2) \subset \mathfrak{g}_2$  preserves only one spinor  $\psi_1$ , and we have  $\mathbb{R}^7 = \mathbb{C}^2 \oplus \mathbb{R}^3$ . The space  $(\Lambda_{27}^3)_{\mathbb{R}^1 \oplus \mathfrak{su}_c(2)}$  is one-dimensional and generated by the 3-form

$$T = \varphi - 7e_{567}.$$

The set  $\text{Tor}_{\mathbb{R}^1 \oplus \mathfrak{su}_c(2)}$  is generated by  $\varphi$  and  $e_{567}$ .

**The subalgebra  $\mathfrak{su}(2) \oplus \mathfrak{su}_c(2) \subset \mathfrak{g}_2$ .**

The subalgebra  $\mathfrak{su}(2) \oplus \mathfrak{su}_c(2)$  is generated by  $P_1, P_2, P_3, Q_1, Q_2, Q_3$  and preserves the spinor  $\psi_1$ , the representation on  $\mathbb{R}^7$  decomposes into  $\mathbb{R}^7 = \mathbb{R}^4 \oplus \mathbb{R}^3$ . The space  $(\Lambda_{27}^3)_{\mathfrak{su}(2) \oplus \mathfrak{su}_c(2)}$  is one-dimensional and generated by the 3-form

$$T = \varphi - 7e_{567},$$

and the set  $\text{Tor}_{\mathfrak{su}(2) \oplus \mathfrak{su}_c(2)}$  is generated by  $\varphi$  and  $e_{567}$ .

**The subalgebra  $\mathfrak{so}(3) \subset \mathfrak{su}(3) \subset \mathfrak{g}_2$ .**

This subalgebra is generated by

$$S_1 := e_{12} - e_{56}, \quad S_2 := e_{13} + e_{24} + e_{35} + e_{46}, \quad S_3 := e_{14} - e_{23} + e_{36} - e_{45}$$

and stabilizes  $\psi_1$  and  $\psi_2$ . We have the further splitting  $\mathbb{R}^7 = \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^1$ , and  $(\Lambda_{27}^3)_{\mathfrak{so}(3)}$  is spanned by three linearly independent 3-forms as follows,

$$\begin{aligned} T_{a,b,c} := & a(-2e_{123} + e_{136} - e_{145} + e_{235} + e_{246} + 2e_{356}) \\ & + b(-2e_{124} - e_{135} - e_{146} + e_{236} - e_{245} + 2e_{456}) \\ & + c(4e_{127} - 3e_{135} + 3e_{146} + 3e_{236} + 3e_{245} + 4e_{347} + 4e_{567}). \end{aligned}$$

$\text{Tor}_{\mathfrak{so}(3)}$  is the union of two hyperplanes in  $\Lambda_1^3 \oplus (\Lambda_{27}^3)_{\mathfrak{so}(3)}$ . The intersection  $\text{Tor}_{\mathfrak{so}(3)} \cap \Lambda_{27}^3$  is two-dimensional and consists of all forms with  $c = 0$ ,  $T_{a,b,0}$ .

**The subalgebra  $\mathfrak{so}_{ir}(3) \subset \mathfrak{g}_2$ .**

The unique irreducible 7-dimensional real representation of  $\mathfrak{so}(3)$  is contained in  $\mathfrak{g}_2$ . Consequently we obtain a maximal subalgebra of dimension three,  $\mathfrak{so}(3)_{ir} \subset \mathfrak{g}_2$ . This algebra preserves only one spinor.  $(\Lambda_{27}^3)_{\mathfrak{so}_{ir}(3)} = 0$  is trivial and  $\text{Tor}_{\mathfrak{so}_{ir}(3)} = \Lambda_1^3$  is generated by  $\varphi$ .

### 3. The characteristic connection of a cocalibrated G<sub>2</sub>-manifold

A G<sub>2</sub>-manifold  $(M^7, g, \varphi)$  is called cocalibrated if the 3-form  $\varphi$  satisfies the differential equation

$$d * \varphi = 0,$$

see [11]. Then there exists a unique connection  $\nabla^c$  preserving the G<sub>2</sub>-structure with totally skew-symmetric torsion (see [13])

$$T^c = \frac{1}{6} (d\varphi, * \varphi) \cdot \varphi - * d\varphi.$$

The torsion form belongs to  $\Lambda_1^3 \oplus \Lambda_{27}^3$ . If the torsion is parallel, the Ricci tensor depends only on the algebraic type of the torsion form  $T^c$ , see [13]. In particular,  $\text{Ric}^{\nabla^c}$  is parallel and divergence free, see [3],

$$\nabla^c \text{Ric}^{\nabla^c} = 0, \quad \text{div}(\text{Ric}^{\nabla^c}) = 0.$$

On all these spaces, there exists a spinor field  $\Psi_1$  satisfying the equations

$$\nabla^c \Psi_1 = 0, \quad T^c \cdot \Psi_1 = a \cdot \Psi_1, \quad \delta(T^c) = 0,$$

where the factor  $a$  depends on the algebraic type of the torsion form.

The subclass of nearly parallel G<sub>2</sub>-structures is given by the condition that the 4-forms  $d\varphi$  and  $*\varphi$  are proportional. In this case the characteristic torsion  $T^c$  is proportional to  $\varphi$  and, consequently, it is automatically parallel,  $\nabla^c T^c = 0$ . Although  $\mathfrak{su}(3)$  preserves the spinors  $\psi_1$  and  $\psi_2$ ,  $\varphi$  acts on these spinors with different eigenvalues, hence no subalgebra of  $\mathfrak{su}(3)$  can occur as holonomy algebra  $\mathfrak{hol}(\nabla^c)$  of the characteristic connection.

**Proposition 3.1.** *Let  $(M^7, g, \varphi)$  be a nearly parallel G<sub>2</sub>-manifold which is not parallel,  $T^c \neq 0$ . Then the holonomy algebra  $\mathfrak{hol}(\nabla^c)$  is not a subalgebra of  $\mathfrak{su}(3)$ . The  $\nabla^c$ -parallel spinor of the nearly parallel G<sub>2</sub>-manifold is unique.*

On the other hand, nearly parallel  $G_2$ -manifolds with characteristic holonomy  $\mathfrak{hol}(\nabla^c) = \mathfrak{su}(2) \oplus \mathfrak{su}_c(2)$  exist. Indeed, consider a 3-Sasakian manifold. By an appropriate rescaling of its metric in the direction of the three-dimensional bundle spanned by the contact forms, one obtains a nearly parallel  $G_2$ -manifold of that type (see [14], Theorem 5.4). The irreducible naturally reductive homogeneous space  $SO(5)/SO_{ir}(3)$  is an example of a nearly parallel  $G_2$ -manifold with characteristic holonomy  $\mathfrak{hol}(\nabla^c) = \mathfrak{so}_{ir}(3)$ .

We now discuss  $G_2$ -manifolds with characteristic torsion in  $\Lambda_{27}^3$  (structures of pure type  $\mathcal{W}_3$ , see [11]). The differential equations characterizing these structures are

$$d * \varphi = 0, \quad (d\varphi, *\varphi) = 0.$$

In this case the formula for the characteristic torsion simplifies,

$$T^c = - * d\varphi.$$

Observe that a cocalibrated  $G_2$ -manifold is of type  $\mathcal{W}_3$  if and only if the Clifford product  $T^c \cdot \Psi = 0$  of the characteristic torsion and the canonical  $\nabla^c$ -parallel spinor vanishes. Since  $T^c \neq 0$  is preserved by the characteristic connection, the holonomy algebra  $\mathfrak{hol}(\nabla^c)$  is a proper subalgebra of  $\mathfrak{g}_2$ . Moreover, the intersection  $\text{Tor}_{\mathfrak{g}} \cap \Lambda_{27}^3 = 0$  is trivial for  $\mathfrak{g} = \mathfrak{su}(3), \mathfrak{su}(2)$  and  $\mathfrak{so}_{ir}(3)$ . The algebraic computations yield now the following result.

**Proposition 3.2.** *Let  $M^7, g, \varphi$  be a  $G_2$ -manifold of type  $\mathcal{W}_3$  with parallel characteristic torsion. If  $\mathfrak{hol}(\nabla^c) = \mathfrak{su}(3), \mathfrak{su}(2)$  or  $\mathfrak{so}_{ir}(3)$ , then  $M^7$  is a parallel  $G_2$ -manifold,  $T^c = 0$ .*

We will study the cocalibrated  $G_2$ -geometries with non-abelian holonomy algebra  $\mathfrak{hol}(\nabla^c)$  case by case.

#### 4. $G_2$ -manifolds with parallel torsion and $\mathfrak{hol}(\nabla^c) = \mathfrak{su}(3)$

A cocalibrated  $G_2$ -manifolds with non-trivial parallel torsion and  $\mathfrak{hol}(\nabla^c) = \mathfrak{su}(3)$  admits a  $\nabla^c$ -parallel vector field  $e_7$  and two  $\nabla^c$ -parallel real spinor fields  $\Psi_1, \Psi_2 = e_7 \cdot \Psi_1$ . Let us introduce the following two globally well defined and  $\nabla^c$ -parallel forms,

$$F := e_{12} + e_{34} + e_{56}, \quad \Sigma := e_{135} - e_{146} - e_{236} - e_{245}$$

and define a  $G_2$ -structure by setting

$$\varphi = \Sigma + F \wedge e_7.$$

The torsion form  $T^c$  acts on these spinor fields with the same or with opposite eigenvalues. First we discuss the case that the torsion form acts on both spinor fields with the same eigenvalue. Up to a scaling of the metric we can assume that  $T^c$  is given by the formula

$$T^c = 2(e_{127} + e_{347} + e_{567}) = 2F \wedge e_7, \quad T^c \cdot \Psi_1 = -6\Psi_1, \quad T^c \cdot \Psi_2 = -6\Psi_2.$$

The equation  $\nabla^c e_7 = 0$  yields directly

$$\nabla_X^g e_7 = X \lrcorner F, \quad de_7 = 2F, \quad T^c = de_7 \wedge e_7.$$

Consequently, the tuple  $(M^7, g, e_7, F)$  is a Sasakian manifold and  $T^c$  is the characteristic torsion of this contact structure, see [13]. Consider the  $\nabla^c$ -parallel spinors  $\Psi_1, \Psi_2$ . Using the special algebraic formula for the torsion we compute that both spinors are solutions of the equation

$$\nabla_X^g \Psi = -\frac{1}{2} X \cdot \Psi - g(X, e_7) \cdot e_7 \cdot \Psi.$$

In the paper [15] we discussed the integrability conditions for this system. In the notations of this paper, the spinors  $\Psi_1, \Psi_2$  are Sasakian quasi-Killing spinors of type  $(-1/2, -1)$ . Then  $(M^7, g, e_7, F)$  has to be a  $\eta$ -Einstein Sasakian manifold with Ricci tensor

$$\text{Ric}^g = 10 \cdot g - 4 \cdot e_7 \otimes e_7.$$

The 3-form  $\Sigma$  can be interpreted in contact geometry, too. It is a horizontal,  $\nabla^c$ -parallel and belongs to the  $\Lambda_2^3(\mathbb{R}^6)$ -component in the decomposition of  $\Lambda^3(\mathbb{R}^6)$  under the action of the group  $U(3)$ , see [4]. Let us fix now a simply-connected  $\eta$ -Einstein Sasakian manifold of the prescribed type. Then there exist two Sasakian quasi-Killing spinors of type  $(-1/2, -1)$  (see [15], Theorem 6.3). Moreover, we can reconstruct the  $G_2$ -structure as well as the  $\nabla^c$ -parallel spinor fields. Finally we obtain the following result.

**Theorem 4.1.** *Any cocalibrated  $G_2$ -manifold such that the characteristic torsion acts on both  $\nabla^c$ -parallel spinors by the same eigenvalue and*

$$\nabla^c T^c = 0, \quad T^c \neq 0, \quad \text{hol}(\nabla^c) = \mathfrak{su}(3)$$

*holds is homothetic to an  $\eta$ -Einstein Sasakian manifold. Its Ricci tensor is given by the formula*

$$\text{Ric}^g = 10 \cdot g - 4 \cdot e_7 \otimes e_7.$$

*Conversely, a simply-connected  $\eta$ -Einstein Sasakian manifold with Ricci tensor  $\text{Ric}^g = 10 \cdot g - 4 \cdot e_7 \otimes e_7$  admits a cocalibrated  $G_2$ -structure with parallel characteristic torsion and characteristic holonomy contained in  $\mathfrak{su}(3)$ .*

The paper [7] is an introduction to the geometry of  $\eta$ -Einstein Sasakian manifolds.

Next we investigate the case that the torsion form acts on the parallel spinor fields with opposite eigenvalues. Then we have

$$T^c = e_{135} - e_{146} - e_{236} - e_{245} = \Sigma, \quad T^c \cdot \Psi_1 = -4 \Psi_1, \quad T^c \cdot \Psi_2 = 4 \Psi_2,$$

and the scalar curvature is  $\text{Scal}^g = 30$ . Since  $T^c$  does not depend on  $e_7$  and  $\nabla^c e_7 = 0$ , we conclude that  $e_7$  is parallel with respect to the Levi-Civita connection,  $\nabla^g e_7 = 0$ . The manifold  $M^7$  splits isometrically into  $M^7 = X^6 \times \mathbb{R}^1$ , where  $X^6$  is an almost complex manifold with Kähler form  $F$  and characteristic torsion  $T^c$ . This torsion form belongs to the  $\Lambda_2^3(\mathbb{R}^6)$ -component in the decomposition of  $\Lambda^3(\mathbb{R}^6)$  under the action of the group  $U(3)$ , see [4]. Consequently,  $X^6$  is a nearly Kähler (non-Kählerian) 6-manifold. Vice versa, we can reconstruct the 7-dimensional  $G_2$ -structure out of the 6-dimensional nearly Kähler structure by the formula

$$\varphi = \Sigma + F \wedge e_7,$$

where  $F$  is the Kähler form and  $\Sigma$  the parallel 3-form of type  $\Lambda_2^3$  of  $X^6$ . Let us summarize the result.

**Theorem 4.2.** *A complete, simply-connected cocalibrated  $G_2$ -manifold such that the characteristic torsion acts on  $\nabla^c$ -parallel spinors by opposite eigenvalues and*

$$\nabla^c T^c = 0, \quad T^c \neq 0, \quad \text{hol}(\nabla^c) = \mathfrak{su}(3)$$

*holds is isometric to the product of a nearly Kähler 6-manifold by  $\mathbb{R}$ . Conversely, any such product admits a cocalibrated  $G_2$ -structure with parallel torsion and holonomy contained in  $\mathfrak{su}(3)$ .*

### 5. $G_2$ -manifolds with parallel torsion and $\mathfrak{hol}(\nabla^c) = \mathfrak{u}(2)$

In this case the following forms are globally well defined and  $\nabla^c$ -parallel,

$$e_7, \quad \Omega_1 := e_{12} + e_{34}, \quad \Omega_2 := e_{56}, \quad \Sigma := e_{135} - e_{146} - e_{236} - e_{245}.$$

The  $G_2$ -structure as well as the admissible torsion forms are given by the formulas

$$\begin{aligned} \varphi &= \Sigma + \Omega_1 \wedge e_7 + \Omega_2 \wedge e_7, \\ T^c &= -(a + b - c)\Sigma + (2a + c)\Omega_1 \wedge e_7 + (4b + c)\Omega_2 \wedge e_7, \end{aligned}$$

where one of the conditions  $a + b = c$  or  $4(a + b) = -3c$  is satisfied. Using Proposition 4.2 of the paper [1] we compute the differentials of these forms.

**Lemma 5.1.**

$$\begin{aligned} de_7 &= (2a + c)\Omega_1 + (4b + c)\Omega_2, \\ d\Omega_1 &= 2(a + b - c)(e_7 \lrcorner \Sigma) = 2d\Omega_2, \\ d\Sigma &= -4(a + b - c)\Omega_1 \wedge \Omega_2 + (4a + 4b + 3c)(*\Sigma), \\ d*\Sigma &= 0, \quad d(e_7 \lrcorner \Sigma) = (4a + 4b + 3c)\Sigma \wedge e_7. \end{aligned}$$

First we study the case  $4a + 4b + 3c = 0, c \neq 0$ . The torsion form is given by

$$T^c = \frac{7}{4}c\Sigma + (2a + c)(\Omega_1 - 2\Omega_2) \wedge e_7$$

and it acts on the spinors  $\Psi_1$  and  $\Psi_2$  with opposite eigenvalues  $\pm 7c$ . We compute the scalar curvature,

$$\text{Scal}^g = -12a^2 - 12ac + \frac{711}{8}c^2.$$

The formulas for the differentials of the globally defined forms can be simplified,

$$\begin{aligned} de_7 &= (2a + c)(\Omega_1 - 2\Omega_2), \quad d\Omega_1 = -\frac{7}{2}c(e_7 \lrcorner \Sigma) = 2d\Omega_2, \\ d*\Sigma &= 0, \quad d(e_7 \lrcorner \Sigma) = 0, \quad d\Sigma = 7c\Omega_1 \wedge \Omega_2. \end{aligned}$$

In particular, all possible Lie derivatives vanish

$$\mathcal{L}_{e_7}\Omega_1 = 0, \quad \mathcal{L}_{e_7}\Omega_2 = 0, \quad \mathcal{L}_{e_7}\Sigma = 0, \quad \mathcal{L}_{e_7}(e_7 \lrcorner \Sigma) = 0, \quad \mathcal{L}_{e_7}T^c = 0.$$

Let us discuss the regular case, i.e. we assume that  $e_7$  induces a free action of the group  $S^1$ . Then  $\pi : M^7 \rightarrow \tilde{X}^6$  is a principal fiber bundle over a smooth manifold  $\tilde{X}^6$ . Moreover, on this manifold there exist differential forms  $\tilde{\Omega}_1, \tilde{\Omega}_2$  and  $\tilde{\Sigma}$  such that

$$\tilde{T}^c = \frac{7}{4}c\tilde{\Sigma}, \quad *\tilde{\Sigma} = -e_7 \lrcorner \Sigma$$

holds, where  $*$  denotes the Hodge operator of  $\tilde{X}^6$ . We introduce the form

$$\tilde{\Omega} := \tilde{\Omega}_1 + \tilde{\Omega}_2.$$

Then we obtain

$$d\tilde{\Omega} = 3\frac{7}{4}c*\tilde{\Sigma} = 3*\tilde{T}^c.$$

The torsion form  $\tilde{T}^c \neq 0$  is of type  $\Lambda_2^3$  in the sense of almost Hermitian geometry on  $\tilde{X}^6$ . The last equation shows that  $\tilde{X}^6$  is a nearly Kähler manifold (see [4], section 4.2.) with reduced characteristic holonomy  $\mathfrak{hol}(\tilde{\nabla}^c) = \mathfrak{u}(2) \subset \mathfrak{su}(3)$ . Then  $\tilde{X}^6$  is isomorphic to the projective space  $\mathbb{CP}^3$  or to the flag manifold  $\mathbb{F}(1, 2)$  equipped with their standard nearly Kähler structure coming from the twistor construction, see [5]. The form  $\tilde{\Omega}_1 - 2\tilde{\Omega}_2$  is their standard Kähler form. Conversely, if  $\tilde{X}^6 = \mathbb{CP}^3, \mathbb{F}(1, 2)$  is



nearly Kähler with reduced characteristic holonomy, then the forms  $\tilde{\Omega} = \tilde{\Omega}_1 + \tilde{\Omega}_2$  and  $\tilde{\Sigma}$  exist and  $d(\tilde{\Omega}_1 - 2\tilde{\Omega}_2) = 0$  holds. The equation

$$de_7 = (2a + c)(\tilde{\Omega}_1 - 2\tilde{\Omega}_2)$$

defines—under the obvious integral condition for the cohomology class  $(2a + c)(\tilde{\Omega}_1 - 2\tilde{\Omega}_2)$ —a  $S^1$ -principal bundle  $\pi : M^7 \rightarrow \tilde{X}^6$  together with a connection. Finally,  $M^7$  admits a  $G_2$ -structure

$$\varphi = \pi^*(\tilde{\Sigma}) + \pi^*(\tilde{\Omega}_1 + \tilde{\Omega}_2) \wedge e_7$$

with parallel characteristic torsion form

$$T^c = \frac{7}{4}c\pi^*(\tilde{\Sigma}) + (2a + c)\pi^*(\tilde{\Omega}_1 - 2\tilde{\Omega}_2) \wedge e_7.$$

All together we classified this type of regular  $G_2$ -manifolds.

**Theorem 5.1.** *Let  $(M^7, g, \varphi)$  be a complete, cocalibrated  $G_2$ -manifold such that*

$$\nabla^c T^c = 0, \quad \mathfrak{hol}(\nabla^c) = \mathfrak{u}(2)$$

*and suppose that  $T^c$  acts with opposite eigenvalues  $\pm 7c \neq 0$  on the  $\nabla^c$ -parallel spinors  $\Psi_1, \Psi_2$ . Moreover, suppose that  $M^7$  is regular. Then  $M^7$  is a principal  $S^1$ -bundle and a Riemannian submersion over the projective space  $\mathbb{CP}^3$  or the flag manifold  $\mathbb{F}(1, 2)$  equipped with their standard nearly Kähler structure coming from the twistor construction. The Chern class of the fibration  $\pi : M^7 \rightarrow \mathbb{CP}^3, \mathbb{F}(1, 2)$  is proportional to the Kähler form. Conversely, any of these fibrations admits a  $G_2$ -structure with parallel characteristic torsion and characteristic holonomy contained in  $\mathfrak{u}(2)$ .*

Consider now the case that  $a + b = c$ . The torsion form is given by

$$T^c = (2a + c)\Omega_1 \wedge e_7 + (5c - 4a)\Omega_2 \wedge e_7$$

and it acts on the spinors  $\Psi_1$  and  $\Psi_2$  with eigenvalue  $-7c$ . Again, we may simplify the formulas for the derivatives,

$$\begin{aligned} de_7 &= (2a + c)\Omega_1 + (5c - 4a)\Omega_2, & d\Omega_1 &= d\Omega_2 = 0, \\ d*\Sigma &= 0, & d(e_7 \lrcorner *\Sigma) &= 7c\Sigma \wedge e_7, & d\Sigma &= 7c(*\Sigma). \end{aligned}$$

In particular, the Lie derivatives are given by

$$\mathcal{L}_{e_7}\Omega_1 = 0, \quad \mathcal{L}_{e_7}\Omega_2 = 0, \quad \mathcal{L}_{e_7}\Sigma = 7c(e_7 \lrcorner *\Sigma), \quad \mathcal{L}_{e_7}(e_7 \lrcorner *\Sigma) = -7c\Sigma.$$

The tangent bundle of  $M^7$  splits into two complex bundles and one real bundle,

$$TM^7 = E_1 \oplus E_2 \oplus \mathbb{R} \cdot e_7,$$

where  $E_1$  is spanned by  $\{e_1, e_2, e_3, e_4\}$  and  $E_2$  is spanned by  $\{e_5, e_6\}$ . The characteristic connection preserves this splitting. Using the formula for the characteristic torsion, we see that the bundles  $E_1 \oplus \mathbb{R}^1 \cdot e_7$  and  $E_2 \oplus \mathbb{R}^1 \cdot e_7$  are preserved by the Levi-Civita connection  $\nabla^g$ ,

$$\nabla^g(E_1 \oplus \mathbb{R} \cdot e_7) \subset E_1 \oplus \mathbb{R}^1 \cdot e_7, \quad \nabla^g(E_2 \oplus \mathbb{R}^1 \cdot e_7) \subset E_2 \oplus \mathbb{R}^1 \cdot e_7.$$

We compute the Ricci tensor as well as the scalar curvature:

$$\begin{aligned} \text{Ric}^{\nabla^c} &= (-4a^2 + 10ac + 6c^2)\text{Id}_{E_1} \oplus (-16a^2 + 12ac + 10c^2)\text{Id}_{E_2}, \\ \text{Ric}^g &= (-2a^2 + 12ac + \frac{13}{2}c^2)\text{Id}_{E_1} \oplus (-8a^2 - 8ac + \frac{45}{2}c^2)\text{Id}_{E_2} \\ &\quad \oplus (12a^2 - 16ac + \frac{27}{2}c^2) \cdot e_7, \\ \text{Scal}^g &= -12a^2 + 16ac + \frac{169}{2}c^2, \quad \text{Scal}^{\nabla^c} = -48a^2 + 64ac + 44c^2. \end{aligned}$$

For a regular structure, the orbit space  $\pi : M^7 \longrightarrow \tilde{X}^6$  admits a Riemannian metric  $\tilde{g}$ , two closed forms  $\tilde{\Omega}_1, \tilde{\Omega}_2$  and a splitting of the tangent bundle,

$$T\tilde{X}^6 = \tilde{E}_1 \oplus \tilde{E}_2.$$

The Levi-Civita connection of  $\tilde{X}^6$  preserves this splitting. Consequently, the universal covering of  $\tilde{X}^6$  splits into the product of two Kähler-Einstein manifolds  $(\tilde{Y}_1, \tilde{g}, \tilde{\Omega}_1)$  and  $(\tilde{Y}_2, \tilde{g}, \tilde{\Omega}_2)$ . The fibers of the Riemannian submersion  $\pi : M^7 \longrightarrow \tilde{X}^6$  are totally geodesic and the O'Neill tensor is given by

$$A_X Y = \frac{1}{2} \text{pr}_{E_2}[X, Y] = -\frac{1}{2} T^c(X, Y, *).$$

We apply formula (9.36c) of [6] and compute the Ricci tensor of  $\tilde{X}^6$

$$\tilde{\text{Ric}}^{\tilde{g}} = 7c(2a + c)\text{Id}_{\tilde{E}_1} \oplus 7c(5c - 4a)\text{Id}_{\tilde{E}_2}.$$

Denote by  $\tilde{S}_i$  the scalar curvature of  $\tilde{Y}_i$  for  $i = 1, 2$ . The sum  $\tilde{S} = \tilde{S}_1 + \tilde{S}_2$  is the scalar curvature of  $\tilde{X}^6$ , and we have

$$\tilde{S}_1 = 28c(2a + c), \quad \tilde{S}_2 = 14c(5c - 4a), \quad \tilde{S} = 7 \cdot 14 \cdot c^2 > 0.$$

By inverting these expression, we may express the parameters  $a, c$  by the scalar curvatures,

$$a = \frac{5\tilde{S}_1 - 2\tilde{S}_2}{28\sqrt{2\tilde{S}}}, \quad c = \frac{\sqrt{\tilde{S}}}{7\sqrt{2}}.$$

The differential  $de_7$  is given by

$$de_7 = (2a + c)\Omega_1 + (5c - 4a)\Omega_2 = \frac{1}{\sqrt{2\tilde{S}}} \left( \frac{1}{2} \tilde{S}_1 \tilde{\Omega}_1 + \tilde{S}_2 \tilde{\Omega}_2 \right) = \frac{\sqrt{2}}{\sqrt{\tilde{S}}} \tilde{\text{Ric}}^{\tilde{g}}.$$

The form  $\Sigma$  is a section in a line bundle. Indeed, let us introduce the complex-valued form  $\sigma := \Sigma + i(e_7 \lrcorner * \Sigma)$ . Then we obtain

$$\mathcal{L}_{e_7} \sigma = -7i c \sigma.$$

Denote by  $L$  the length of the closed integral curves of  $e_7$ .  $\sigma$  is periodic along the integral curve, i.e.  $7cL = 2k\pi$ . Since

$$\sigma(m \cdot e^{2\pi ti}) = e^{-7cLti} \cdot \sigma(m) = e^{-2\pi kti} \cdot \sigma(m),$$

the map  $\sigma$  is a section  $\Lambda_2^3(\tilde{X}^6) \otimes G_k$ , where  $G_k = M^7 \times_{S^1} \mathbb{C}$  is the associated bundle defined by the  $S^1$ -representation  $z \longrightarrow z^k$ . The section  $\sigma$  is parallel. The complex-valued 1-form

$$\frac{2\pi}{L} e_7 \cdot i : TM^7 \longrightarrow \mathbb{R} \cdot i$$

is the connection form in the bundle  $\pi : M^7 \longrightarrow \tilde{X}^6$ . The Chern class of this principal bundle is given by

$$c_1^* = -\frac{1}{2\pi k} \tilde{\text{Ric}}^{\tilde{g}} = -\frac{1}{k} c_1(\tilde{X}^6) .$$

Consequently, the curvature of the bundle  $\Lambda_2^3(\tilde{X}^6) \otimes G_k$  vanishes automatically. Moreover, in the non-simply-connected case the holonomy of the flat bundle  $\Lambda_2^3(\tilde{X}^6) \otimes G_k$  is trivial.

**Theorem 5.2.** *Let  $(M^7, g, \varphi)$  be a complete, cocalibrated G<sub>2</sub>-manifold such that*

$$\nabla^c T^c = 0, \quad \text{hol}(\nabla^c) = \mathfrak{u}(2)$$

*and suppose that  $T^c$  acts with eigenvalue  $-7c \neq 0$  on the  $\nabla^c$ -parallel spinors  $\Psi_1, \Psi_2$ . Moreover, suppose that  $M^7$  is regular. Then  $M^7$  is a principal S<sup>1</sup>-bundle and a Riemannian submersion over a Kähler manifold  $\tilde{X}^6$ . This manifold has the following properties:*

- (1) *The universal covering of  $\tilde{X}^6$  splits into a 4-dimensional Kähler-Einstein manifold and a 2-dimensional surface with constant curvature.*
- (2) *The scalar curvature  $\tilde{S} = \tilde{S}_1 + \tilde{S}_2 > 0$  is positive.*
- (3) *The Kähler forms  $\tilde{\Omega}_1$  and  $\tilde{\Omega}_2$  are globally defined on  $\tilde{X}^6$ .*

*The bundle  $\pi : M^7 \longrightarrow \tilde{X}^6$  is defined by a connection form. Its curvature is proportional to the Ricci form of  $\tilde{X}^6$ . Finally, the flat bundle  $\Lambda_2^3(\tilde{X}^6) \otimes G_k$  admits a parallel section. Conversely, any S<sup>1</sup>-bundle resulting from this construction admits a cocalibrated G<sub>2</sub>-structure such that the characteristic torsion is parallel and the characteristic holonomy is contained in  $\mathfrak{u}(2)$ .*

**Example 5.1.** Let  $\tilde{Y}_1$  be a simply-connected Kähler-Einstein manifold with negative scalar curvature  $\tilde{S}_1 = -1$ , for example a hypersurface of degree  $d \geq 5$  in  $\mathbb{CP}^3$ . For the second factor we choose the round sphere normalized by the condition  $\tilde{S}_2 = +2$ . Then the product  $\tilde{X}^6 = \tilde{Y}_1 \times \tilde{Y}_2$  is simply-connected and the S<sup>1</sup>-bundle defined by the Ricci form admits a cocalibrated G<sub>2</sub>-structure with parallel torsion. Since the product  $\tilde{X}^6 = \tilde{Y}_1 \times \tilde{Y}_2$  is simply-connected, the flat bundle  $\Lambda_2^3(\tilde{X}^6) \otimes G_1$  admits a parallel section  $\sigma$ .

Finally we study the case  $c = 0$ . The G<sub>2</sub>-manifold is of pure type  $\mathcal{W}_3$ . In this case the 3-form  $\Sigma$  projects onto  $\tilde{X}^6$  and defines a parallel form in  $\Lambda_2^3(\tilde{X}^6)$ . On the other hand, the curvature of the bundle  $\pi : M^7 \rightarrow \tilde{X}^6$  is proportional to  $\tilde{\Omega}_1 - 2\tilde{\Omega}_2$ .

**Theorem 5.3.** *Let  $(M^7, g, \varphi)$  be a complete G<sub>2</sub>-manifold of pure type  $\mathcal{W}_3$  such that*

$$\nabla^c T^c = 0, \quad T^c \neq 0, \quad \text{hol}(\nabla^c) = \mathfrak{u}(2).$$

*Moreover, suppose that  $M^7$  is regular. Then  $M^7$  is a principal S<sup>1</sup>-bundle and a Riemannian submersion over a Ricci-flat Kähler manifold  $\tilde{X}^6$ . This manifold has the following properties:*

- (1) *The universal covering of  $\tilde{X}^6$  splits into a 4-dimensional Ricci-flat Kähler manifold and the 2-dimensional flat space  $\mathbb{R}^2$ .*
- (2) *The Kähler forms  $\tilde{\Omega}_1$  and  $\tilde{\Omega}_2$  are globally defined on  $\tilde{X}^6$ .*
- (3) *There exists a parallel form  $\Sigma \in \Lambda_2^3(\tilde{X}^6)$ .*

The bundle  $\pi : M^7 \longrightarrow \tilde{X}^6$  is defined by a connection form. Its curvature is proportional to the form

$$\tilde{\Omega}_1 - 2\tilde{\Omega}_2.$$

Conversely, any  $S^1$ -bundle resulting from this construction admits a  $G_2$ -structure of pure type  $\mathcal{W}_3$  such that the characteristic torsion is parallel and the characteristic holonomy is contained in  $\mathfrak{u}(2)$ .

**Example 5.2.** Consider a  $K3$ -surface and denote by  $\tilde{\Omega}_1$  its Kähler form. Then there exist two parallel forms  $\eta_1, \eta_2$  in  $\Lambda_+^2(K3)$  being orthogonal to  $\tilde{\Omega}_1$ . Let  $e_5$  and  $e_6$  be a parallel frame on the torus  $T^2$ . The product  $\tilde{X}^6 = K3 \times T^2$  satisfies the conditions of the latter Theorem. Indeed, we can construct the following parallel form

$$\Sigma = \eta_1 \wedge e_5 + \eta_2 \wedge e_6.$$

Moreover, the cohomology class of  $\tilde{\Omega}_1 - 2\tilde{\Omega}_2$  has to be proportional to an integral class. This implies the condition that  $\tilde{\Omega}_1/\text{vol}(T^2) \in H^2(K3; \mathbb{Q})$  is a rational cohomology class.

## 6. $G_2$ -manifolds with parallel torsion and $\mathfrak{hol}(\nabla^c) = \mathfrak{su}(2)$

We briefly discuss the structure of simply-connected, complete, cocalibrated  $G_2$ -manifolds with parallel characteristic torsion and  $\mathfrak{hol}(\nabla^c) = \mathfrak{su}(2)$ . The tangent bundle splits into the sum of two bundles preserved by the characteristic connection,

$$TM^7 = E_1 \oplus E_2.$$

In our notation, the three-dimensional subbundle  $E_2$  is spanned by  $\{e_5, e_6, e_7\}$ . Moreover, the following forms are globally defined and  $\nabla^c$ -parallel,

$$e_5, \quad e_6, \quad e_7, \quad \Omega_1 := e_{12} + e_{34}, \quad \Omega_2 := e_{14} + e_{23}, \quad \Omega_3 := e_{13} - e_{24}.$$

The  $G_2$ -structure is given by the formula

$$\varphi = \Omega_1 \wedge e_7 - \Omega_2 \wedge e_6 + \Omega_3 \wedge e_5 + e_{567}.$$

Basically there are three algebraic types of torsion forms. If  $T^c = e_{567}$ , then  $M^7$  splits into the product  $M^7 = Y^4 \times S^3$  of the sphere  $S^3$  by a simply-connected, complete, Ricci-flat and anti-selfdual manifold  $Y^4$ . The forms  $\Omega_1, \Omega_2, \Omega_3$  are the parallel forms in  $\Lambda_+^2(Y^4)$ . Conversely, any product of that type admits a cocalibrated  $G_2$ -structure with holonomy  $\mathfrak{hol}(\nabla^c) = \mathfrak{su}(2)$ . If  $T^c = \Omega_3 \wedge e_5$ , then  $e_6, e_7$  are  $\nabla^g$ -parallel. The manifold splits into  $M^7 = Y^5 \times \mathbb{R}^2$ . Moreover, we obtain

$$de_5 = \Omega_3, \quad (de_5)^2 \wedge e_5 \neq 0, \quad T^c = de_5 \wedge e_5.$$

The tuple  $(Y^5, g, e_5, \Omega_3)$  is homothetic to a Sasakian manifold with characteristic torsion  $T^c$  and holonomy  $\mathfrak{su}(2)$ . These spaces have been described completely in [13], Theorem 7.3. and Example 7.4. They are  $\eta$ -Einstein Sasakian manifolds with Ricci tensor  $\text{Ric}^g = (6, 6, 6, 6, 4)$ . Again, we can reconstruct the  $G_2$ -structure of  $Y^5 \times \mathbb{R}^2$  out of the  $\eta$ -Einstein Sasakian structure of  $Y^5$ . The third possibility for the torsion form is

$$T^c = \Omega_1 \wedge e_7 + \Omega_2 \wedge e_6 - \Omega_3 \wedge e_5 - 2e_{567}.$$

A computation of the Ricci tensor yields the following result:

$$\text{Ric}^{\nabla^c} = 3\text{Id}_{E_1} \oplus 0\text{Id}_{E_2}, \quad \text{Ric}^g = \frac{9}{2}\text{Id}_{E_1} \oplus 3\text{Id}_{E_2}.$$

Since  $e_5, e_6, e_7$  are  $\nabla^c$ -parallel, the Killing vector fields define a locally free isometric action of the group  $SU(2)$  on  $M^7$ . We identify the Lie algebra of  $SU(2)$  with these Killing

vector fields. In the regular case the G<sub>2</sub>-manifold is a principal SU(2)-bundle  $M^7 \rightarrow Y^4$  over a smooth Riemannian four-manifold. The vector valued 1-form  $Z : TM^7 \rightarrow \mathfrak{su}(2)$  defined by

$$Z := e_5 \otimes e_5 + e_6 \otimes e_6 + e_7 \otimes e_7$$

is a connection form. The formulas

$$de_5 = -\Omega_3 - 2e_{67}, \quad de_6 = \Omega_2 + 2e_{57}, \quad de_7 = \Omega_1 - 2e_{56},$$

express the curvature of the connection  $Z$ ,

$$\Omega_Z = \Omega_1 \otimes e_7 + \Omega_2 \otimes e_6 - \Omega_3 \otimes e_5.$$

Consequently,  $M^7$  is a SU(2)-instanton bundle over  $Y^4$  and the selfdual curvature is parallel.

### 7. G<sub>2</sub>-manifolds with parallel torsion and $\mathfrak{hol}(\nabla^c) = \mathfrak{so}(3)$

A G<sub>2</sub>-manifold with characteristic holonomy  $\mathfrak{hol}(\nabla^c) = \mathfrak{so}(3)$  admits two  $\nabla^c$ -parallel spinor fields  $\Psi_1, \Psi_2$ , a  $\nabla^c$ -parallel vector field  $e_7$  and a  $\nabla^c$ -parallel 2-form  $e_{12} + e_{34} + e_{56}$ . Moreover, the representation splits into  $\mathbb{R}^7 = \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^1$ . Consequently, the condition  $\nabla^c T^c = 0$  implies that the manifold is naturally reductive (see [9]). The manifold is a homogeneous space  $G/SO(3)$ , where the Lie algebra  $\mathfrak{g}$  of the 10-dimensional group  $G$  is completely fixed by the torsion and the curvature of the characteristic connection. Indeed, we have  $\mathfrak{g} = \mathfrak{so}(3) \oplus \mathbb{R}^7$  and the bracket is given by the formula

$$[A + X, B + Y] = ([A, B] - R^c(X, Y)) + (A \cdot Y - B \cdot X - T^c(X, Y)).$$

The family of admissible torsion forms depends on three parameters,

$$\begin{aligned} T_{a,b,c} &= d \cdot \varphi + a(-2e_{123} + e_{136} - e_{145} + e_{235} + e_{246} + 2e_{356}) \\ &+ b(-2e_{124} - e_{135} - e_{146} + e_{236} - e_{245} + 2e_{456}) \\ &+ c(4e_{127} - 3e_{135} + 3e_{146} + 3e_{236} + 3e_{245} + 4e_{347} + 4e_{567}), \end{aligned}$$

where either  $d = 3c$  or  $d = -4c$ . If  $d = 3c$ , the torsion form acts on  $\Psi_1, \Psi_2$  with the same eigenvalue. For  $d = -4c$  it acts on these spinors with opposite eigenvalues. The curvature operator  $R^c : \Lambda^2(\mathbb{R}^7) \rightarrow \mathfrak{so}(3)$  is invariant. This gives a five-dimensional space parameterized by  $\{x, y, z, u, v\}$ ,

$$\begin{aligned} R^c &= R_1^c \otimes S_1 + R_2^c \otimes S_2 + R_3^c \otimes S_3, \\ R_1^c &= (x - y)(e_{12} - e_{56}) + (x + y)(e_{16} + e_{25}) + 2z(e_{26} - e_{15}) + 2u e_{37} + 2v e_{47}, \\ R_2^c &= x(e_{13} + e_{35}) + z(e_{14} + e_{23} + e_{36} + e_{45}) + v(e_{17} + e_{57}) - y(e_{24} + e_{46}) \\ &\quad - u(e_{27} + e_{67}), \\ R_3^c &= z(e_{13} - e_{24} - e_{35} + e_{46}) + y(e_{45} - e_{14}) + u(e_{57} - e_{17}) + x(e_{36} - e_{23}) \\ &\quad + v(e_{67} - e_{27}). \end{aligned}$$

Since the characteristic torsion is parallel, the curvature tensor  $R^c(X, Y, U, V)$  is symmetric with respect to the pairs  $(X, Y)$  and  $(U, V)$ , see [13]. This implies directly that  $u = v = z = 0$  and  $y = -x$ . The curvature operator is proportional to the projection  $\text{pr} : \Lambda^2(\mathbb{R}^7) \rightarrow \mathfrak{so}(3)$  onto the Lie subalgebra,

$$R^c = x(2S_1 \otimes S_1 + S_2 \otimes S_2 + S_3 \otimes S_3).$$

The pair  $(T^c, R^c)$  has to satisfy the Bianchi identity. In particular,  $T^c \cdot T^c + R^c$  is a scalar in the Clifford algebra. This equation has two solutions, namely:

$$\begin{aligned} d &= -4c, & x &= a^2 + b^2 - 49c^2, \\ d &= 3c, & a &= b = 0, & 2x &= -49c^2. \end{aligned}$$

If  $d = 3c \neq 0$ , then the parameters  $a = b = 0$  vanish. The torsion form depends only on the parameter  $c$  and can be written in the simpler form

$$T^c = 7c(e_{12} + e_{34} + e_{56}) \wedge e_7, \quad R^c = -\frac{49}{2}c^2(2S_1 \otimes S_1 + S_2 \otimes S_2 + S_3 \otimes S_3).$$

A computation of the Ricci tensor yields the following result

$$\text{Ric}^g(e_i) = \frac{5}{2}49c^2e_i \quad \text{for } i = 1, \dots, 6, \quad \text{Ric}^g(e_7) = \frac{3}{2}49c^2e_7.$$

In particular, the Riemannian Ricci tensor is positive definite and, consequently,  $M^7$  is compact. One easily identifies the group  $G$ : It is the group  $G = \text{SO}(5)$  with the standard embedding of  $\text{SO}(3)$ . The corresponding naturally reductive space is the Stiefel manifold  $M^7 = \text{SO}(5)/\text{SO}(3)$ .

**Theorem 7.1.** *A simply-connected, complete, cocalibrated  $G_2$ -manifold with characteristic holonomy  $\mathfrak{hol}(\nabla^c) = \mathfrak{so}(3)$  such that  $T^c$  acts with the same eigenvalue on the parallel spinors  $\Psi_1, \Psi_2$  is isometric to the Stiefel manifold  $\text{SO}(5)/\text{SO}(3)$ . The metric is a Riemannian submersion over the Grassmanian manifold  $G_{5,2}$ .*

If  $d = -4c$ , the torsion form does not depend on  $e_7$ ,  $e_7 \lrcorner T^c = 0$ . Since  $e_7$  is  $\nabla^c$ -parallel, the vector field is parallel with respect to the Levi-Civita connection, too. A complete, simply-connected  $G_2$ -manifold of that type splits into the Riemannian product  $Y^6 \times \mathbb{R}^1$ , where  $Y^6$  is an almost Hermitian manifold of Gray-Hervella-type  $\mathcal{W}_1 \oplus \mathcal{W}_3$  with characteristic holonomy  $\mathfrak{so}(3) \subset \mathfrak{su}(3)$ .  $Y^4$  is completely defined by the torsion form and this family of almost Hermitian 6-manifolds has been studied in [4], Theorem 4.6 as well as in [17]. Let us summarize the result.

**Theorem 7.2.** *A simply-connected, complete, cocalibrated  $G_2$ -manifold with characteristic holonomy  $\mathfrak{hol}(\nabla^c) = \mathfrak{so}(3)$  such that  $T^c$  acts with opposite eigenvalues on the parallel spinors  $\Psi_1, \Psi_2$  splits into the Riemannian product  $Y^6 \times \mathbb{R}^1$ , where  $Y^6$  is an almost Hermitian manifold of Gray-Hervella-type  $\mathcal{W}_1 \oplus \mathcal{W}_3$  with characteristic holonomy  $\mathfrak{so}(3) \subset \mathfrak{su}(3)$ .*

## 8. $G_2$ -manifolds with parallel torsion and $\mathfrak{hol}(\nabla^c) = \mathfrak{so}_{ir}(3)$

Since  $(\Lambda_{27}^3)_{\mathfrak{so}_{ir}(3)} = 0$  is trivial, any cocalibrated  $G_2$ -manifold with parallel characteristic torsion and  $\mathfrak{hol}(\nabla^c) = \mathfrak{so}_{ir}(3)$  is nearly parallel. Moreover, the curvature tensor  $R^c$  is  $\nabla^c$ -parallel, see [9]. There exists only one  $\mathfrak{so}_{ir}(3)$ -invariant curvature operator  $R^c : \Lambda^2(\mathbb{R}^7) \rightarrow \mathfrak{so}_{ir}(3)$ , namely the projection onto the subalgebra  $\mathfrak{so}_{ir}(3) \subset \mathfrak{so}(7) = \Lambda^2(\mathbb{R}^7)$ . Consequently, the characteristic torsion and the curvature operator are uniquely defined. On the other side, consider the embedding of  $\text{SO}(3)$  into  $\text{SO}(5)$  given by the 5-dimensional irreducible  $\text{SO}(3)$ -representation. Then the naturally reductive space  $\text{SO}(5)/\text{SO}_{ir}(3)$  admits a nearly parallel  $G_2$ -structure, see [14]. Finally we obtain a complete classification in this case.

**Theorem 8.1.** *A complete, simply-connected and cocalibrated  $G_2$ -manifold with parallel characteristic torsion and  $\mathfrak{hol}(\nabla^c) = \mathfrak{so}_{ir}(3)$  is isometric to  $\text{SO}(5)/\text{SO}_{ir}(3)$ .*

9. G<sub>2</sub>-manifolds with parallel torsion and  $\mathfrak{hol}(\nabla^c) = \mathfrak{su}_c(2)$ 

In this section we prove the following uniqueness result.

**Theorem 9.1.** *There exists a unique simply-connected, complete, cocalibrated G<sub>2</sub>-manifold with*

$$\nabla^c T^c = 0, \quad \mathfrak{hol}(\nabla^c) = \mathfrak{su}_c(2).$$

*The manifold is homogeneous naturally reductive.*

*Proof.* The  $\mathfrak{su}_c(2)$ -representation  $\mathbb{R}^7 = \mathbb{C}^2 \oplus \mathbb{R}^3$  splits into the sum of two irreducible representations. Then any G<sub>2</sub>-structure with parallel torsion and holonomy  $\mathfrak{su}_c(2)$  is naturally reductive,  $\nabla^c T^c = 0, \nabla^c R^c = 0$ , see [9]. The torsion forms belong to  $\text{Tor}_{\mathfrak{su}_c(2)}$  and are parameterized by two parameters,  $T^c = a \cdot \varphi + b \cdot e_{567}$ . The curvature operator  $R^c : \Lambda^2(\mathbb{R}^7) \rightarrow \mathfrak{su}_c(2) \subset \Lambda^2(\mathbb{R}^7)$  is symmetric and  $\mathfrak{su}_c(2)$ -invariant. Consequently, the curvature operator is proportional to the projection onto the Lie subalgebra  $\mathfrak{su}_c(2)$ ,

$$R^c = p(Q_1 \otimes Q_1 + Q_2 \otimes Q_2 + Q_3 \otimes Q_3).$$

$Q_1, Q_2$  and  $Q_3$  denote the basis of the Lie algebra  $\mathfrak{su}_c(2)$  introduced before. The pair  $(T^c, R^c)$  satisfies the Bianchi identity if and only if  $(T^c)^2 + R^c$  is a scalar in the Clifford algebra, see [16]. There are two solutions of this algebraic equation,

$$a = 0, p = 0, \quad \text{and} \quad p = a^2, 5a + b = 0.$$

The case  $p = 0, a = 0$  defines a flat structure,  $\mathfrak{hol}(\nabla^c) = 0$ . The second case yields a unique naturally reductive, cocalibrated G<sub>2</sub>-manifold with parallel torsion and holonomy  $\mathfrak{hol}(\nabla^c) = \mathfrak{su}_c(2)$ . The Lie algebra  $\mathfrak{g}$  of the 10-dimensional automorphism group is given by  $\mathfrak{g} = \mathfrak{su}_c(2) \oplus \mathbb{R}^7$  with the bracket

$$[A + X, B + Y] = ([A, B] - R^c(X, Y)) + (A \cdot Y - B \cdot X - T^c(X, Y)).$$

It turns out that  $\mathfrak{g}$  is perfect,  $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ . The adjoint representation  $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$  is a faithful representation of  $\mathfrak{g}$ . The Lie algebra  $\mathfrak{g}$  contains a 7-dimensional nilpotent radical  $\mathfrak{r}$ . It is generated by  $\mathfrak{r} = \text{Lin}(e_1, e_2, e_3, e_4, e_5 - Q_2, e_6 + Q_1, e_7 + Q_3)$ . Moreover,  $[\mathfrak{r}, \mathfrak{r}] = \text{Lin}(e_5 - Q_2, e_6 + Q_1, e_7 + Q_3)$  is three-dimensional and abelian.  $\mathfrak{g}/\mathfrak{r} = \mathfrak{su}_c(2)$  is isomorphic to the holonomy algebra.  $\square$

The case of  $\mathfrak{hol}(\nabla^c) = \mathbb{R}^1 \oplus \mathfrak{su}_c(2)$  is similar. The admissible torsion forms are again  $T^c = a \cdot \varphi + b \cdot e_{567}$  and the  $(\mathbb{R}^1 \oplus \mathfrak{su}_c(2))$ -invariant operators  $R^c : \Lambda^2(\mathbb{R}^7) \rightarrow \mathbb{R}^1 \oplus \mathfrak{su}_c(2)$  are parameterized by three parameters,

$$\begin{aligned} R^c &= p \cdot (e_{34} \otimes Q_3 - e_{14} \otimes Q_1 - e_{23} \otimes Q_1 - e_{13} \otimes Q_2 + e_{24} \otimes Q_2 + e_{12} \otimes Q_3) \\ &+ q \cdot (e_{56} \otimes Q_3 - e_{57} \otimes Q_1 - e_{67} \otimes Q_2) + r \cdot (e_{13} + e_{24}) \otimes P_1. \end{aligned}$$

Since  $R^c$  is symmetric, we have  $q = -2p$ . The curvature operator simplifies,

$$R^c = p(Q_1 \otimes Q_1 + Q_2 \otimes Q_2 + Q_3 \otimes Q_3) + r P_1 \otimes P_1.$$

$(T^c)^2 + R^c$  is a scalar in the Clifford algebra if and only if the following relations between the parameters hold,

$$p = -\frac{1}{2}a(3a + b), \quad r = \frac{3}{2}a(5a + b).$$

If  $5a + b \neq 0$  and  $3a + b \neq 0$ , then the holonomy of the characteristic connection is the full Lie algebra  $\mathbb{R}^1 \oplus \mathfrak{su}_c(2)$ . All together we classified simply-connected, complete, cocalibrated G<sub>2</sub>-manifolds with parallel characteristic torsion and holonomy  $\mathbb{R}^1 \oplus \mathfrak{su}_c(2)$ .

The spaces are naturally reductive. Up to a scaling, the family depends on one parameter.

**Remark 9.1.** If  $b = 0$ , then  $M^7 = N(1,1)$  is a nearly parallel  $G_2$ -manifold with an 11-dimensional automorphism group. The automorphism group is isomorphic to  $SU(3) \times SU(2)$  and the space appears in the classification of all nearly parallel  $G_2$ -manifolds with a large automorphism group, see [14].

#### 10. $G_2$ -manifolds with parallel torsion and $\mathfrak{hol}(\nabla^c) = \mathfrak{su}(2) \oplus \mathfrak{su}_c(2)$

**Example 10.1.** Starting with a 3-Sasakian manifold and rescaling again its metric along the three-dimensional bundle spanned by  $e_5, e_6, e_7$ , one obtains a family  $(M^7, g_s, \varphi_s)$  of cocalibrated  $G_2$ -manifold such that

$$d *_s \varphi_s = 0, \quad T_s^c = \left(\frac{2}{s} - 10s\right) e_{567}^* + 2s\varphi_s, \quad \nabla^c T_s^c = 0.$$

The necessary computations proving these properties are contained in [14], Theorem 5.4. The characteristic connection preserves the splitting of the tangent bundle and, consequently, its holonomy is  $\mathfrak{hol}(\nabla^c) = \mathfrak{su}(2) \oplus \mathfrak{su}_c(2)$ . If  $s = 1/\sqrt{5}$ , the structure is nearly parallel. Since  $(T_s^c, \varphi_s) = 4s + 2/s > 0$ , these structures are never of pure type  $\mathcal{W}_3$ . In the parametrization of the admissible torsion forms  $T^c = a\varphi + be_{567}$  the family realizes the following curve in the  $\{a, b\}$ -plane,

$$a = 2s, \quad b = \frac{2}{s} - 10s, \quad 5a^2 + ab = 4.$$

First we investigate naturally reductive  $G_2$ -manifolds with holonomy  $\mathfrak{hol}(\nabla^c) = \mathfrak{su}(2) \oplus \mathfrak{su}_c(2)$ . The characteristic torsion and the curvature operator depend on two parameters,  $T^c = a \cdot \varphi + b \cdot e_{567}$  and

$$R^c = p(Q_1 \otimes Q_1 + Q_2 \otimes Q_2 + Q_3 \otimes Q_3) + r(P_1 \otimes P_1 + P_2 \otimes P_2 + P_3 \otimes P_3).$$

The pair  $(T^c, R^c)$  satisfies the Bianchi identity if and only if

$$r = -\frac{a}{2}(5a + b), \quad p = -\frac{a}{2}(3a + b).$$

Consequently, we obtain (up to scaling) a one-parameter family of naturally reductive homogeneous  $G_2$ -manifolds with  $\mathfrak{hol}(\nabla^c) = \mathfrak{su}(2) \oplus \mathfrak{su}_c(2)$ .

**Remark 10.1.** For  $b = 0$ , the manifold  $M^7$  is nearly parallel and has a 13-dimensional automorphism group, the squashed 7-sphere. It appears again in the classification in [14].

A classification of all cocalibrated  $G_2$ -manifolds with parallel torsion and holonomy  $\mathfrak{su}(2) \oplus \mathfrak{su}_c(2)$  seems to be inaccessible. Nevertheless we can discuss the geometry of such manifolds and describe some particular cases. The tangent bundle splits into the sum of two bundles preserved by the characteristic connection,

$$TM^7 = E_1 \oplus E_2.$$

In our notation, the three-dimensional subbundle  $E_2$  is spanned by  $\{e_5, e_6, e_7\}$ . The parallel torsion form of a cocalibrated  $G_2$ -manifold depends on two parameters,

$$T = a\varphi + be_{567}.$$



The  $\nabla^c$ -parallel spinor field  $\Psi_1$  satisfies the following differential equations

$$\nabla_X^g \Psi_1 = -\frac{3a}{4} X \cdot \Psi_1 \quad \text{for } X \in E_1, \quad \nabla_V^g \Psi_1 = -\frac{3a+b}{4} V \cdot \Psi_1 \quad \text{for } V \in E_2.$$

The Ricci tensor depends only on T and can be computed explicitly

$$\begin{aligned} \text{Ric}^{\nabla^c} &= (12a^2 + 3ab) \text{Id}_{E_1} \oplus (12a^2 + 4ab) \text{Id}_{E_2}, \\ \text{Ric}^g &= \left(\frac{27}{2}a^2 + 3ab\right) \text{Id}_{E_1} \oplus (13a^2 + 4ab + \frac{1}{2}(a+b)^2) \text{Id}_{E_2}. \end{aligned}$$

Since  $\nabla^c$  preserves the splitting  $TM^7 = E_1 \oplus E_2$ , the algebraic formula for the torsion yields that  $E_2$  is an involutive subbundle. Moreover, the leaves are totally geodesic. We prove now that every leaf of this distribution is a 3-dimensional Riemannian manifold of constant sectional curvature. The result is a consequence of the following formulas.

**Lemma 10.1.** *Let  $X$  be a vector field in  $E_1$  and  $V$  be a vector field in  $E_2$ . Then we have*

$$\begin{aligned} g(\nabla_V^g V, \nabla_X^g X) &= 0, \quad g(\nabla_X^g \nabla_V^g V, X) = 0, \\ R^g(X, V, V, X) &= \frac{1}{4} \|T(X, V, *)\|^2, \quad \sum_{i=1}^4 R^g(e_i, V, V, e_i) = a^2 \|V\|^2. \end{aligned}$$

*Proof.*  $\nabla^c$  preserves the splitting  $TM^7 = E_1 \oplus E_2$  and the torsion is totally skew-symmetric. Then we have

$$\begin{aligned} 0 &= g(\nabla_V^c V, \nabla_X^c X) = g(\nabla_V^g V, \nabla_X^g X) = -g(\nabla_X^g \nabla_V^g V, X) + X(g(\nabla_V^g V, X)) \\ &= -g(\nabla_X^g \nabla_V^g V, X) + X(g(\nabla_V^c V, X)) = -g(\nabla_X^g \nabla_V^g V, X). \end{aligned}$$

We compute the formula for the curvature tensor  $R^g(X, V, V, X)$  in a similar way. Alternatively, one can apply formula (9.29b) in [6].  $\square$

We already know the Ricci tensor of  $M^7$  and that every leaf of the distribution is totally geodesic. Then the preceding Lemma yields a formula for the Ricci tensor of the leaves,

$$(12a^2 + 4ab + \frac{1}{2}(a+b)^2) \text{Id} = \frac{1}{2}(5a+b)^2 \text{Id}.$$

**Proposition 10.1.** *Every leaf is a 3-dimensional Riemannian manifold of constant sectional curvature*

$$k = \frac{1}{4}(5a+b)^2 \geq 0.$$

*If  $M^7$  is complete, every maximal leaf of the distribution  $E_2$  is isometric to a complete space form of non-negative sectional curvature.*

If the space of leaves is a smooth manifold,  $\pi : M^7 \rightarrow Y^4$  is a Riemannian submersion with totally geodesic fibers. The relevant tensor relating the geometry of  $M^7$  and  $Y^4$  is

$$A_X Y = \frac{1}{2} \text{pr}_{E_2} [X, Y] = -\frac{1}{2} T^c(X, Y, *).$$

Remark that in our case the tensor  $A : \Lambda^2(E_1) \rightarrow E_2$  vanishes on  $\Lambda_-^2(E_1)$ . Moreover, if  $a \neq 0$ , then  $A : \Lambda_+^2(E_1) \rightarrow E_2$  is an isomorphism. We apply the formula (9.36c) of [6] and we compute the Ricci tensor of the space of leaves.

**Proposition 10.2.** *The space of leaves  $Y^4$  is an Einstein space. Its Ricci tensor is given by  $3a(5a + b)\text{Id}_{TY^4}$ .*

If  $a = 0$ , then the holonomy is reduced to  $\mathfrak{su}(2)$ . Indeed, in this case, the Levi-Civita connection preserves the splitting of the tangent bundle. The universal covering of  $M^7$  splits into the Riemannian product. The three-dimensional factor is a sphere  $S^3$ . The four-dimensional factor is anti-selfdual and Ricci-flat. The vector fields  $e_5, e_6, e_7$  become  $\nabla^c$ -parallel and globally defined. For example, we obtain

$$\nabla_X^c e_5 = \nabla_X^{S^3} e_5 - \frac{b}{2} e_6 \wedge e_7(X, *) = 0.$$

Consequently, the holonomy is  $\mathfrak{hol}(\nabla^c) = \mathfrak{su}(2)$ . The second interesting case  $b = 0$  corresponds to nearly parallel  $G_2$ -manifolds. According to Proposition 3.1, Theorem 8.1 and Theorem 9.1, any nearly parallel  $G_2$ -manifold different from  $\text{SO}(5)/\text{SO}_{ir}(3)$  and  $N(1, 1) = (\text{SU}(3) \times \text{SU}(2))/(\text{S}^1 \times \text{SU}(2))$  has characteristic holonomy  $\mathfrak{su}(2) \oplus \mathfrak{su}_c(2)$  or  $\mathfrak{g}_2$ . At present, only few examples of nearly parallel  $G_2$ -manifolds are known; hence, a complete classification of the case  $b = 0$  is inaccessible.

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